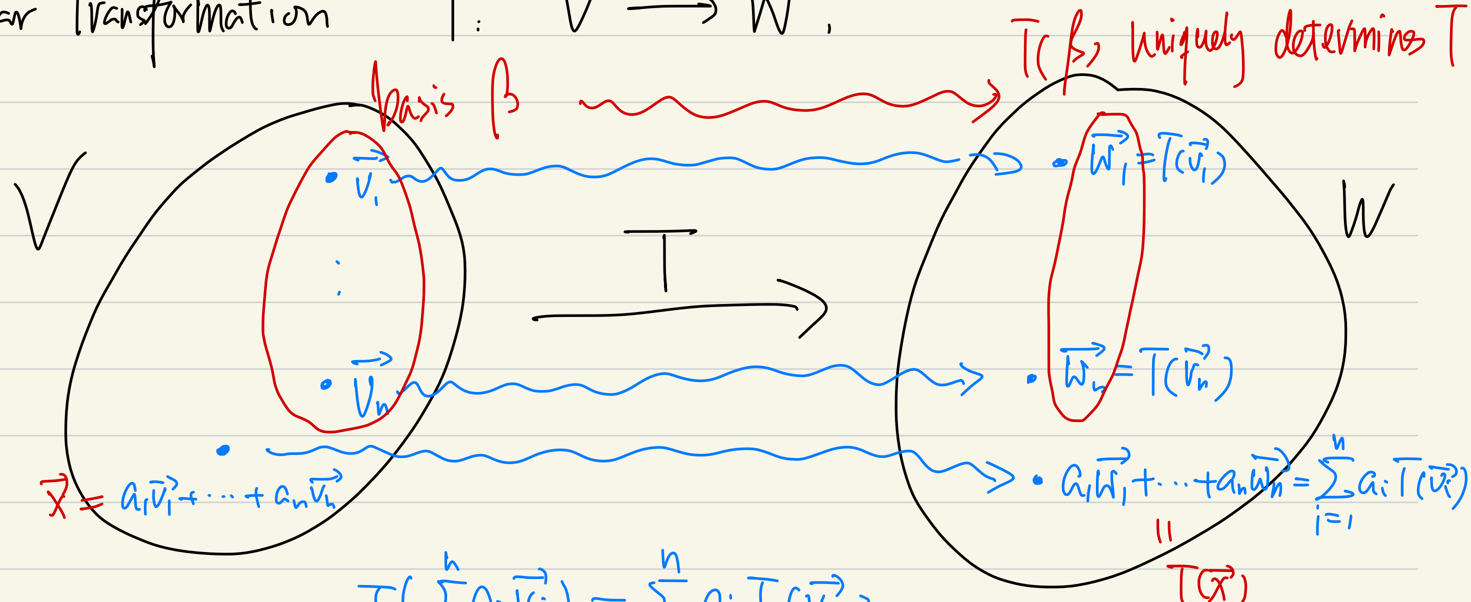
Linear Transformation $T: V \to W$,



$$T\left(\sum_{i=1}^{n} a_i \vec{v}_i\right) = \sum_{i=1}^{n} a_i T(\vec{v}_i)$$

§ Matrix representations of linear transformations.

Def: An ordered basis for a finite dim. vector space V is a basis for V with a specific order

e.g. $\{e_1, e_2\} \neq \{e_2, e_1\}$ as ordered basis.

Def: V vector space. $\beta = \{\vec{u_1}, \cdots, \vec{u_n}\}$ ordered basis for V.
Then $\forall \vec{x} \in V$, $\exists$ unique $a_1, \cdots, a_n \in F$

s.t. $$\vec{x} = \sum_{i=1}^{n} a_i \vec{u_i}$$

The coordinate vector of $\vec{x}$ relative to β, denoted $[x]_\beta := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in F^n$

Def: Let $T: V \to W$ linear transformation.

dim $V = n$ $\quad \beta = \{\vec{v}_1, \cdots \vec{v}_n\}$ o. b. for V.

dim $W = m$ $\quad \gamma = \{\vec{w}_1, \cdots, \vec{w}_m\}$ o. b. for W.

For each $1 \le j \le n$, $\exists\, a_{ij} \in F$, $1 \le i \le m$. s.t.

$$T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij} \vec{w}_i .$$

Define Matrix representation of T in the ordered basis β and γ:

$$[T]_\beta^\gamma = A := (a_{ij})_{\substack{1 \le i \le m, \\ 1 \le j \le n}} \in M_{m \times n}$$

- In particular, if $V = W$ and $\beta = \gamma$, write $[T]_\beta := [T]_\beta^\beta$.

More explicitly, $T(v_1) = a_{11}\vec{w_1} + \cdots + a_{m1}\vec{w_m} \iff [T(v_1)]_\gamma = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$

$\cdots$

$T(v_n) = a_{1n}\vec{w_1} + \cdots + a_{mn}\vec{w_m} \iff [T(v_n)]_\gamma = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$

$$[T]_\beta^\gamma = A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} = \left.\underbrace{\begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix}}_{n}\right\} m$$

$[T(v_1)]_\gamma$ $\qquad$ $[T(v_n)]_\gamma$

Examples :

- For the left multiplication. $L_A : F^n \longrightarrow F^m$ with matrix $A \in M_{m\times n}(F)$.

  We have $[L_A]_\beta^\gamma = A$ in the standard basis β and γ of F^n, F^m, resp.

pf:
$$[L_A]_\beta^\gamma = \begin{pmatrix} | & & | \\ [L_A(\vec{e}_1)]_\gamma , & \cdots & [L_A(\vec{e}_n)]_\gamma \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ [A\vec{e}_1]_\gamma & \cdots & [A(e_n)]_\gamma \\ | & & | \end{pmatrix} = A .$$

first col. of A.   nth col of A

- For $T: P_n(\mathbb{R}) \longrightarrow P_{n-1}(\mathbb{R})$ defined by $T(f(x)) = f'(x)$.

Using the standard order bases $\beta = \{1, x, \cdots, x^n\}$ and $\gamma = \{1, x, \cdots, x^{n-1}\}$

Note that $T(1) = 0$, $T(x) = 1$, $T(x^2) = 2x$, $\cdots$, $T(x^n) = nx^{n-1}$.

$$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} | & & | \\ [T(1)]_\gamma & \cdots & [T(x_n)]_\gamma \\ | & & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 3 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

- $T: M_{2\times 2}(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$ linear.

$$A \rightsquigarrow A^T + 2A.$$

Let $\beta = \left\{ \overset{\vec{v}_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \overset{\vec{v}_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{\vec{v}_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \overset{\vec{v}_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$ o.b. for $M_{2\times 2}(\mathbb{R})$.

$$T(\beta) = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \right\}$$

Then $$[T]_\beta = \begin{pmatrix} | & & | \\ [T(v_1)]_\beta & \cdots & [T(v_4)]_\beta \\ | & & | \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

#.

Thm: $T: V \to W$ linear transformation. β and γ are ordered basis for V, W resp.

Then for any $\vec{v} \in V$:

$$[T(\vec{v})]_\gamma = [T]_\beta^\gamma \cdot [\vec{v}]_\beta$$

Pf: Let $\vec{v} = \sum_{j=1}^{n} c_j \vec{v}_j \iff [\vec{v}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix}$

then
$$\begin{aligned} T(\vec{v}) &= \sum_{j=1}^{n} c_j T(\vec{v}_j) \\ &= \sum_{j=1}^{n} c_j \sum_{i=1}^{m} a_{ij} \vec{w}_i \\ &= \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} c_j \cdot a_{ij}\right)}_{j^{th} \text{ coord of } [T(\vec{v})]_\gamma} \vec{w}_i \end{aligned}$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d] & W \arrow[d] \\ F^n \arrow[r, "{[T]_\beta^\gamma}"] & F^m \end{tikzcd}$$

$$\begin{tikzcd} \vec{v} \arrow[r, squiggly] \arrow[d, squiggly] & \vec{w} \arrow[d, squiggly] \\ {[\vec{v}]_\beta} \arrow[r, squiggly] & {[\vec{w}]_\gamma} \end{tikzcd}$$

$\square$

§. Composition of linear transformations and matrix multiplication.

Theorem. Let V, W, Z be vector spaces, and $T: V \to W$ and $U: W \to Z$ linear transf.

(i). Then the composition $UT: V \to Z$ is linear.

(ii). If V, W, Z have ordered bases α, β, γ resp. then

$$\boxed{[UT]_\alpha^\gamma = [U]_\beta^\gamma \cdot [T]_\alpha^\beta}$$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "{[\cdot]_\alpha}"'] \arrow[rr, bend left, "UT"] & W \arrow[r, "U"] \arrow[d, "{[\cdot]_\beta}"'] & Z \arrow[d, "{[\cdot]_\gamma}"] \\
F^n \arrow[r, "{[T]_\alpha^\beta}"] \arrow[rr, bend right, "{[UT]_\alpha^\gamma}"'] & F^m \arrow[r, "{[U]_\beta^\gamma}"] & F^p.
\end{tikzcd}$$

Pf: (i): $\forall \vec{x}, \vec{y} \in V$, $c \in F$: $UT(\vec{x}+\vec{y}) = U(T(\vec{x})+T(\vec{y})) = UT(\vec{x}) + UT(\vec{y})$

$UT(c\vec{x}) = U(cT(\vec{x})) = cUT(\vec{x})$.

(ii). Suppose $\alpha = \{\vec{v_1}, \cdots, \vec{v_n}\}$. $\beta = \{\vec{w_1}, \cdots, \vec{w_m}\}$, $\gamma = \{\vec{z_1}, \cdots, \vec{z_p}\}$

Let $[U]_\beta^\gamma = A = (a_{ik})_{\substack{1\le i\le p \\ 1\le k\le n}} \in M_{p\times m}(F)$. $\Leftrightarrow$ $U(\vec{w_k}) = \sum_{i=1}^{p} a_{ik}\vec{z_i}$ for $1 \le k \le m$

$[T]_\alpha^\beta = B = (b_{kj})_{\substack{1\le k\le n \\ 1\le j\le m}} \in M_{m\times n}(F)$ $\Leftrightarrow$ $T(\vec{v_j}) = \sum_{k=1}^{m} b_{kj}\vec{w_k}$ for $1 \le j \le n$

Then $$UT(\vec{v_j}) = U\left(\sum_{k=1}^{m} b_{kj}\vec{w_k}\right)$$

$$= \sum_{k=1}^{m} b_{kj}\, U(\vec{w_k})$$

$$= \sum_{k=1}^{m} b_{kj}\left(\sum_{i=1}^{p} a_{ik}\vec{z_i}\right)$$

$$= \sum_{i=1}^{p}\left(\underbrace{\sum_{k=1}^{m} b_{kj}\cdot a_{ik}}_{(i,j)^{th}\text{ entry of } AB}\right)\vec{z_i}.$$

$$\Rightarrow [UT]_\alpha^\gamma = \left(\left.[UT(\vec{v_1})]_\gamma\right| \cdots \left|[UT(\vec{v_n})]_\gamma\right.\right) = AB = [U]_\beta^\gamma \cdot [T]_\alpha^\beta$$

□

Example: Consider $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by $T(f(x)) := f'(x)$.

and $U: P_{n-1}(\mathbb{R}) \to P_n(\mathbb{R})$ defined by $U(f(x)) := \int_0^x f(t)\,dt$.

In the standard basis β, γ of $P_n(\mathbb{R})$ and $P_{n-1}(\mathbb{R})$ resp,

$\beta = \{1, x, \cdots, x^n\}$

$\gamma = \{1, x, \cdots, x^{n-1}\}$

We have

$$[T]_\beta^\gamma = \underbrace{\begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots \\ & & \vdots & & 0 \\ 0 & 0 & 0 & & n \end{bmatrix}}_{n+1} \Big] n$$

$$[U]_\gamma^\beta = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1/2 & & \vdots \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & & 1/n \end{bmatrix}$$

$U(1) = x$ $\quad$ $U(x) = \frac{1}{2}x^2$ $\quad$ $U(x^{n-1}) = \frac{1}{n}x^n = 0\cdot 1 + 0\cdot x + \cdots + \frac{1}{n}\cdot x^n$

Calculus $\Rightarrow$ $I = TU : P_{n-1}(\mathbb{R}) \longrightarrow P_{n-1}(\mathbb{R})$

Check: $[TU]_\gamma = [T]_\beta^\gamma \cdot [U]_\gamma^\beta$

$$= \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots 0 & & 1 \end{pmatrix}$$

$$= I_n$$

(Ex: Check UT)

§ Invertibility and Isomorphism

Def: Let $T: V \longrightarrow W$ linear transformation.

We say T is invertible if it is bijective.

$\Longleftrightarrow \exists\, T^{-1}: W \longrightarrow V$ s.t. $T \circ T^{-1} = I_W$ and $T^{-1} \circ T = I_V$.

Prop: The inverse $T^{-1}: W \longrightarrow V$ is linear.

Pf: Given $\vec{y_1}, \vec{y_2} \in W$, $c \in F$. Assume $T(\vec{x_i}) = \vec{y_i}$ $\Rightarrow$ $T(\vec{x_1} + \vec{x_2}) = \vec{y_1} + \vec{y_2}$, $T(c\vec{x_i}) = c\vec{y_i}$

then $T^{-1}(\vec{y_i}) = \vec{x_i}$, $T^{-1}(\vec{y_1} + \vec{y_2}) = \vec{x_1} + \vec{x_2}$, $T^{-1}(c\vec{y_i}) = c\vec{x_i}$ $\Rightarrow$ linear

$\square$

<u>Example</u>: • Let $A \in M_{n\times n}(F)$ be invertible.

Then the left mult. by A. $L_A : F^n \to F^n$

$\vec{x} \rightsquigarrow A\vec{x}$

is invertible and its inverse is given by $L_{A^{-1}}$.

• If $T: V \to W$ and $U: W \to Z$ are invertible

then $UT: V \to Z$ is also invertible

and $(UT)^{-1} = T^{-1}U^{-1}$

Prop: Suppose $T: V \to W$ is invertible. Then $\dim V < \infty \iff \dim W < \infty$.

and in this case, $\dim V = \dim W$.

Pf: Suppose $\dim V = n < +\infty$, and $\beta = \{\vec{x}_1, \cdots, \vec{x}_n\}$ is a basis for V.

Then $W = R(T) = \text{Span}(T(\beta))$, So $\dim W \leq n = \dim V < +\infty$

Applying the same argument to $T^{-1}: W \to V$ shows that $\dim W < +\infty$ implies $\dim V \leq \dim W < +\infty$.

Note: rank-nullity thm $\Rightarrow \underbrace{\dim N(T)}_{\overset{\text{inj}}{=} 0} + \underbrace{\dim R(T)}_{\overset{\text{surj}}{=} \dim W} = \dim V$

Prop: Let V and W be finite-dim vector spaces with ordered bases β and γ, resp. Let $T: V \to W$ linear.

Then T is invertible iff $[T]_\beta^\gamma$ is invertible.

Furthermore, $\boxed{[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}}$

Pf: $(\Rightarrow)$: Suppose T is invertible, then $\dim V = \dim W = n$.

Since $TT^{-1} = I_W$. $\quad [TT^{-1}]_\gamma = [T]_\beta^\gamma \cdot [T^{-1}]_\gamma^\beta$

$W \xrightarrow{T^{-1}} V \xrightarrow{I} W$ $\quad = [I_W]_\gamma = I_n$

Similiarly, $T^{-1}T = I_V$. $\Rightarrow$ $I_n = [T^{-1}]_\gamma^\beta \cdot [T]_\beta^\gamma$

Hence $[T]_\beta^\gamma$ is invertible matrix and $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$.

$(\Leftarrow)$: Conversely, Suppose $[T]_\beta^\gamma$ is invertible

Since $\dim V = \dim W$, we only need to show that T is injective.

Suppose $T(\vec{x_1}) = T(\vec{x_2})$. then $[T(\vec{x_1})]_\gamma = [T(\vec{x_2})]_\gamma$

$$\Rightarrow [T]_\beta^\gamma [\vec{x_1}]_\beta = [T]_\beta^\gamma \cdot [\vec{x_2}]_\beta$$

$\Rightarrow [x_1]_\beta = [x_2]_\beta$ $\Rightarrow$ $\vec{x_1} = \vec{x_2}$.

Def: Vector spaces V and W are isomorphic

if $\exists$ invertible linear transformation $T: V \longrightarrow W$

In this case, T is called an isomorphism from V to W.

Thm: Let V and W be finite-dim vector spaces.

Then V is isomorphic to W iff $\dim V = \dim W$.

pf: ($\Rightarrow$): If $T: V \to W$ is invertible, then $\dim V = \dim W$ (rank-nullity)

($\Leftarrow$): Suppose $\dim V = \dim W = n$.

Let $\beta = \{\vec{v_1}, \cdots, \vec{v_n}\}$, $\gamma = \{\vec{w_1}, \cdots, \vec{w_n}\}$ basis for V and W.

Then $\exists$ linear $T: V \to W$ s.t. $T(\vec{v_i}) = \vec{w_i}$ for $i = 1, \cdots, n$.

T is clearly invertible $\square$

Corollary: Let V be a vector space over F. Then V is isomorphic to F^n iff $\dim V = n$.

Def: Let β be an ordered basis for an n-dim vector space V over F.

Then $\phi_\beta : V \longrightarrow F^n$ is an isomorphism,

$\vec{v} \rightsquigarrow [\vec{v}]_\beta$

Called the "standard representation of V with respect to β".